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# **Chapter #1:**

## Section #1.1

### Classification by Type

* + If is used then it is ODE.
  + If is used then it is PDE.

### Classification by Order

Second Order First Order

* Order is always a positive integer (Order).
* Do not confuse Order with Degree (Power) of a derivative.
* Always choose the highest order.

### Classification by Degree

* Degree is always taken of the Highest Order.

### Classification by Linearity

* A differential equation is said to be Linear if it is in the form of the given equation.
* The dependent variable (in this case) ‘y’ and all its derivatives are of the 1st degree.
* The coefficients of depend at most on the independent variable (in this case) x.

# **Chapter #2:**

## Section #2.2 (Separable Equations):

If a differential equation is in form given above, then it can solved by the following method:

#### Example #1

## Section #2.3 (Linear Equations)

If a differential equation is in form given above, then it can solved by the following method:

#### Example #1

## Section #2.4 (Exact & Non-Exact Equations)

### Exact Differential Equations

If there is an equation that has a variable depending on two independent variables

Then its derivative will be something like

However when then

A function is known to be an Exact Differential Equation when

How to Solve:

#### Example #1

### Non-Exact Differential Equations

What if an equation is not in exact form? Meaning

Then we multiply with an integrating factor

#### Example #2

## Section #2.5 (Solutions by Substitution)

### Homogeneous Equations

If a function possesses the property for some real number then *f* is said to be a homogeneous function of degree

Is said to be homogeneous if both coefficient functions M and N are homogeneous functions of the SAME degree.

In addition, if M and N are homogeneous functions of degree, we can also write

This will turn the equation into a separable equation which can be solved much easier.

#### Example #1

Although either of the indicated substitutions can be used for every homogeneous differential equation, in practice we try whenever the function is simpler than. Also it could happen that after using one substitution, we may encounter integrals that are difficult or impossible to evaluate in closed form; switching substitutions may result in an easier problem.

### Bernoulli’s Equation

If a differential equation is in form given above, where n is any real number, then it can solved by the following method:

#### Example #2

### Riccati’s Equation

If a differential equation is in form given above, then it can solved by the following method:

#### Example #3

# **Chapter #3:**

## Section #3.1 (Linear Models)

### Growth and Decay Model

The above differential equation is given for a Growth and Decay model, and the way to solve it is using Separable Differential Equations/Linear Differential Equations.

#### Example #1

A culture initially has number of bacteria. At the number of bacteria is measured to be. If the rate of growth is proportional to the number of bacteria present at time t, determine the time necessary for the number of bacteria to triple.

### Newton’s Law of Cooling/Warming

The above differential equation is given for a Cooling/Warming of any object, and the way to solve it is using Separable Differential Equations/Linear Differential Equations.

#### Example #2

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

### LR-Series Circuits

The above equation given for a series circuit containing only a resistor and an inductor. L is the inductance, R is the resistance, E (t) is the voltage supplied, and I is the current. It can easily be solved using Linear Differential Equations.

#### Example #3

A 12-volt battery is connected to a series circuit in which the inductance is 0.5 henry and the resistance is 10 ohms. Determine the current I if the initial current is zero.

### RC-Series Circuit

The above equation given for a series circuit containing only a resistor and a Capacitor. C is the capacitance, R is the resistance, E (t) is the voltage supplied, and I is the current. It can easily be solved using Linear Differential Equations.

#### Example #4

A 100-volt electromotive force is applied to an RC-series circuit in which the resistance is 200 ohms and the capacitance is farad. Find the charge on the capacitor if. Find the current.

## Section #3.2 (Non-Linear Models)

### Logistic Equations

If we wanted to keep the Growth and Decay Model to a certain limit, meaning it would only grow/diminish till a certain point, we will use the logistic equation.

#### Example #1

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days

# **Chapter #4:**

## Section #4.1

### Homogeneous Equations

A linear nth-order differential equation of the form

Is said to be Homogeneous.

However if

Then it is said to be Nonhomogeneous.

### Differential Operators

In calculus differentiation is often denoted by the capital letter D. The symbol D is called the differential operator because it transforms a differentiable function into another function.

In general, we define an nth-order differential operator or polynomial operator to be

Where L is also known as the Linear Operator.

Any linear differential equation can be expressed in terms of the D notation. For example, the differential equation can also be written as.

### Superposition Principle

Let be solutions of the homogeneous nth-order differential equation on an interval I. Then the linear combination, where are arbitrary constants, is also a solution on the interval.

### Linear Dependence/Independence

A set of functions is said to be linearly dependent on an interval I if there exist constants, not all zero, such that

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be linearly independent.

Also a set of n functions is linearly independent on the interval I if no one function is a linear combination of the other functions.

### Wronskian

Suppose each of the functions possesses at least derivatives. The determinant

where the primes denote derivatives, is called the Wronskian of the functions.

If Wronskian = 0 then functions are linearly dependent.

### Fundamental Set of Solutions (Definition)

Any set of n linearly independent solutions of the homogeneous linear nth-order differential equation on an interval I is said to be a fundamental set of solutions on the interval.

## Section #4.2 (Reduction of Order)

Suppose that denotes a nontrivial solution of a Homogeneous Equation and that is defined on an interval. We seek a second solution so that the set consisting of and is linearly independent on.

If and are linearly independent, then their quotient is non constant on —that is, or. The function can be found by substituting into the given differential equation.

#### Example #1

Given that is a solution of on the interval use reduction of order to find a second solution.

If we generalize the steps in the example given above, we can generate a formula for the steps above

This formula only gives the second term, and to create the general solution you need to add the given solution to it.

## Section #4.3 (Homogeneous Linear Equations with Constant Coefficients)

### Case 1 *(b2-4ac > 0)*

Where and are real and distinct roots (Using Superposition Principle to make fundamental set as they are linearly independent)

### Case 2 *(b2-4ac = 0)*

Where and are real and repeated roots (Using Reduction of Order, since we only have 1 solution)

### Case 3 *(b2-4ac < 0)*

Where and are conjugate complex roots (Using Case 1 and Euler’s Formula of)

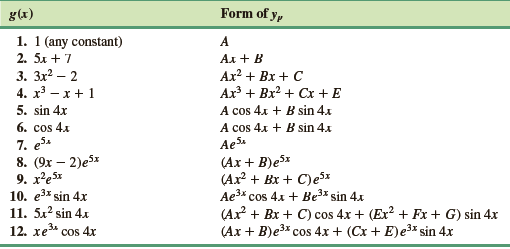
#### Example #1

## Section #4.4 (Undetermined Coefficients—Superposition Approach)

Now we will learn how to solve a nonhomogeneous differential equation.

First we solve the left side as if it were equal to zero, meaning we will do what we did in Section #4.3. The resulting equation would be termed as (Complementary Function). Then we would assume a particular solution for and would find.The final solution would be

How to find



These are just some of the solutions.

**IMPORTANT:** When considering make sure that there are no duplicate terms in and since they would just add up. To avoid this issue make sure to multiply by and try again. Keep multiplying with until you get a new term. Also this method only works when is algebraic, exponential, Sine, and Cosine. It doesn’t work for logarithms, inverse trigonometric, Tangent, Secant, Cosecant, and Cotangent.

#### Example #1

## Section #4.5 (Undetermined Coefficients—Annihilator Approach)

We know that an nth-order differential equation can be written as such

This is very helpful since it can help us annihilate which will affect the left side of the equation and will form a new solution which would include the particular solution. When seeking an annihilator operator, we seek the lowest possible

Some Annihilator Operators:

|  |  |  |
| --- | --- | --- |
|  | **Annihilator Operator** | **Comment** |
|  |  | Where n-1 is the power of x |
|  |  | Where n-1 is the power of x and is the power |
|  |  | Where is the coefficient of in the trigonometric function |
|  |  | ‘’ |

#### Example #1

## Section #4.6 (Variation of Parameters)

Since the methods above were very limited, we can use Variation of Parameters to bypass those limitations. The general idea of Variation of Parameters is that whenever a particular solution is formed, instead of giving it constants we will multiply with variables. Hence the particular solution will look like this

Then we differentiate this and substitute it into the original differential equation. In the end there will exist a system of solution which can be expressed in terms of determinants due to Crammers’ Rule.

The functions are found by integrating the equations given above.

#### Example #1

## Section #4.7 (Cauchy-Euler Equations)

If a differential equation is in form given above, then it can solved by the following method:

### Case 1 (b2-4ac > 0)

Where and are real and distinct roots (Using Superposition Principle to make fundamental set as they are linearly independent)

### Case 2 (b2-4ac = 0)

Where and are real and repeated roots (Using Reduction of Order, since we only have 1 solution)

### Case 3 (b2-4ac < 0)

Where and are conjugate complex roots (Using Case 1 and Euler’s Formula of)

#### Example #1

# **Chapter #11:**

## Section #11.1

### Inner Product

The inner product of any two functions is the integral of their multiple.

### Norm

The norm can be expressed as such

### Orthogonal Functions

### Orthogonal Sets

A set of real-valued functions is said to be orthogonal on an interval [a, b] if

### Orthonormal Sets

If is an orthogonal set of functions on the interval [a, b] with the additional property that for then is said to be an orthonormal set on the interval.

Orthonormal sets can be formed by dividing each function in the orthogonal set by its norm.

### Orthogonal Series Expansion

## Section #11.2 (Fourier Series)

Suppose that f is a function defined on the interval and can be expanded in an orthogonal series consisting of the trigonometric functions

We can use the laws of orthogonality to find.

### Convergence at Discontinuity

Let be piecewise continuous on the interval. Then for all in the interval, the Fourier series of at a point continuity. At a point of discontinuity the Fourier series converges to the average

where are the right and left hand limits of , respectively.

#### Example #1

## Section #11.2 (Fourier Cosine and Sine Series)

### Even and Odd Functions

The Fourier series can be simplified using identities of even and odd functions.

There are some properties of even/odd functions:

1. The product of two even functions is even.
2. The product of two odd functions is even.
3. The product of an even function and an odd function is odd.
4. The sum (difference) of two even functions is even.
5. The sum (difference) of two odd functions is odd.
6. If is even, then
7. If f is odd, then

### Cosine and Sine Series

If f is an even function on then

If f is an odd function on then

### Half-Range Expansions

Whenever we were integrating, we were taking the origin as the midpoint. What if we wanted to find the Fourier series over an interval?

#### Example #1

# **Chapter #12:** Section #12.1 Partial Differential Equations

If we let denote the dependent variable and let denote the independent variables, then the general form of a linear second-order partial differential equation is given by

When the equation is said to be homogeneous otherwise it is nonhomogeneous.

### Separation of Variables

One of the ways to find a solution to a partial differential equation is to suppose

We can easily differentiate this partially and substitute in the equation. The resulting equation would be a function dependent solely on x and a function dependent solely on y, which means each side of the equation must be a constant. Therefore to separate them we use a separation constant.

For the equation

### Case 1

### Case 2

### Case 3

### Superposition Principle

If are solutions of the homogeneous linear partial differential equation, then the linear combination, where are arbitrary constants, is also a solution.

### Classification

The linear second-order partial differential equation

where are real constants, is said to be

* Hyperbolic if
* Parabolic if
* Elliptic if

## Section #12.3 Heat Equation

Consider a thin rod of length with an initial temperature throughout and whose ends are held at temperature zero/are insulated for all time. The temperature in the rod is determined from the boundary-value problem.

Solution:

Using boundary conditions and are trivial solutions, however results in

This formula is only valid for a certain/specific condition. The formula used can be changed depending upon the situation. For example if the ends are not at zero temperature but instead are insulated.

#### Example #1

Suppose heat is lost from the lateral surface of a thin rod of length into a surrounding medium at temperature zero. If the linear law of heat transfer applies, then the heat equation takes on the form.

is a constant. Find the temperature if the initial temperature is throughout and the ends are insulated.

## Section #12.4 Wave Equation

The vertical displacement of the vibrating string of length is determined from

Solution

# **Derivatives**

## Exponential, Logarithmic, and Common Derivatives

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## Trigonometric Derivatives

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## Hyperbolic Derivatives

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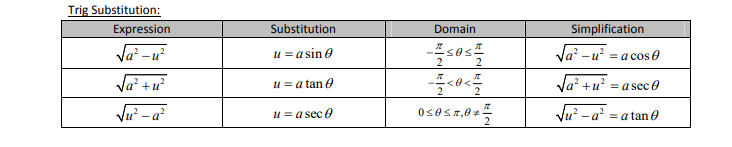
# **Integrals**

## Exponential, Logarithmic, and Common Integrals

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## Trigonometric Integrals

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## Hyperbolic Integrals

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# **Trigonometric Identities**

## Pythagorean Identities

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## Double Angle Formulas

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## Sum and Difference Formulas

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## Product to Sum Formulas

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## Sum to Product Formulas

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